Identifying Unconditional Quantile Impulse Responses with an Application to Growth-at-Risk

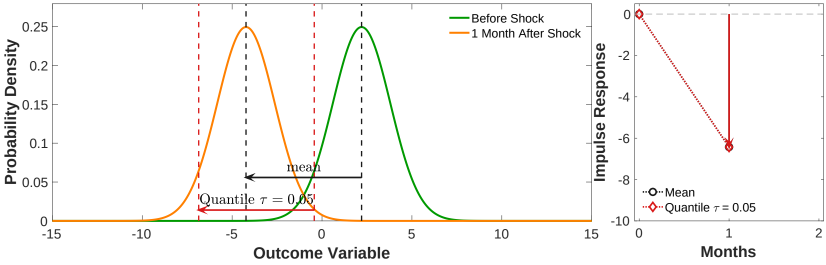
Robert Wojciechowski

Universitat Pompeu Fabra

February 2, 2026

Why Quantile Impulse Responses (QIRs)?

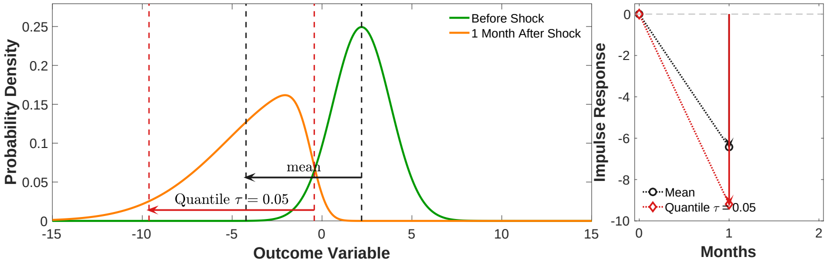
- Shocks can disproportionately affect downside risk $\rightarrow$ small effects in expansions, but cause large losses in downturns.
- Policymaking is tail-sensitive $\rightarrow$ in Sep 2025 the Fed cut rates because "downside risks to employment have risen."
- Mean Impulse Responses identify average effects only.
- QIRs capture how a shock affects the *entire outcome distribution* $\rightarrow$ directly measure the downside risk response.



Illustrating how QIRs describe the effect of a shock on the outcome distribution

Why Quantile Impulse Responses (QIRs)?

- Shocks can disproportionately affect downside risk $\rightarrow$ small effects in expansions, but cause large losses in downturns.
- Policymaking is tail-sensitive $\rightarrow$ in Sep 2025 the Fed cut rates because "downside risks to employment have risen."
- Mean Impulse Responses identify average effects only.
- QIRs capture how a shock affects the *entire outcome distribution* $\rightarrow$ directly measure the downside risk response.



Illustrating how QIRs describe the effect of a shock on the outcome distribution

Unconditional versus Conditional quantiles

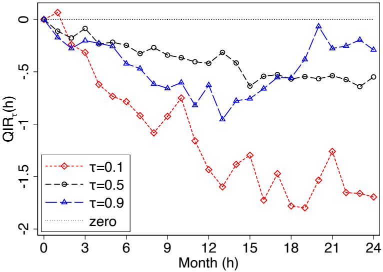
- Unconditional tails $\rightarrow$ extreme outcome periods.
- Conditional tails $\rightarrow$ periods that are extreme relative to what the control variables predict.
- Researchers add controls for causal identification $\rightarrow$ quantile regression then delivers **conditional** quantile effects.
- **Identification** vs **Interpretation** trade-off:
  - No controls $\rightarrow$ effect on **unconditional** quantiles
    clear interpretation, but higher risk of bias
  - Controls $\rightarrow$ effect on **conditional** quantiles
    more plausible identification, but answers a different question
  - A problem not encountered in conditional mean models
    even controls that are statistically independent of the treatment can alter the estimated quantile treatment effects

Generalized Quantile Local Projections (GQLP)

- GQLP identifies **unconditional** QIRs while using controls for identification.
  - Explicitly distinguishes treatment from control variables.
  - Controls serve *only* to isolate the shock.
- GQLP on endogenous treatment and controls identifies the same response as quantile regression on the *true unobserved* structural shock only.
  - Parallel with identification in Local Projections and SVARs.
  - Intuitive interpretation of the QIR:
    $\rightarrow$ *dynamic causal effect of a shock on quantile of outcome.*

Application to Growth-at-Risk

Cumulative losses in U.S. industrial production growth (p.p.) at quantiles $\tau \in 0.1, 0.5, 0.9$ from a 1 s.d. credit risk shock.

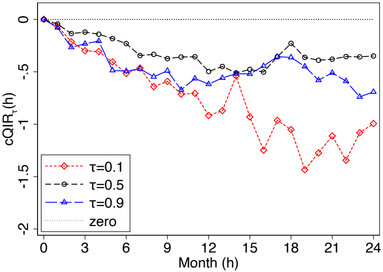


(a) GQLP

Unconditional left tail

↓

Low growth periods



(b) QLP

Conditional left tail

↓

Lower-than-expected growth periods

Contributions to the Literature

Quantile Impulse Responses – *causal identification of unconditional QIRs in the presence of controls.*
(Loria et al. 2025; Chavleishvili and Manganelli 2024; Linnemann and Winkler 2016)

Growth-at-Risk – *causal link between financial risk shocks and unconditional tail risks.*
(Adrian et al. 2022; Adrian et al. 2019; Brownlees and Souza 2021)

Macrofinance – *Quantile Invariance Theorem and simulation algorithm to recover true QIRs from intractable models.*
(Gertler et al. 2019; Bloom 2009)

Notation

GQLP combines *generalized quantile regression* of Powell 2020 with the *local projections* approach of Jordá 2005 in a *potential outcomes for time-series* framework of Rambachan and Shephard 2021.

- $Y_t \in \mathbb{R}^k$; $Y_{i,t}$: scalar outcome; $Y_{j,t}$: scalar treatment.
- $X_t \subset \mathcal{F}_t^Y$: finite history of observables.
- $W_t \in \mathbb{R}^k$: vector of assignments (unobservable shocks).
- Capital letters $\rightarrow$ random variables, lowercase $\rightarrow$ realizations.

Potential Outcomes

- $Y_t(w_{1:t}) \rightarrow$ potential outcomes given a history of shocks.
- $Y_t \equiv Y_t(W_{1:t}) \rightarrow$ observed outcome.
- Shortcut notation for Impulse Response analysis:

$$Y_{i,t+h}(w_j) \equiv Y_{i,t+h}(W_{1:t-1}, W_{1:j-1,t}, w_j, W_{j+1:k,t}, W_{t+1:t+h})$$

  $\rightarrow$ Observed outcome: $Y_{i,t+h} \equiv Y_{i,t+h}(W_{j,t})$.
- Potential outcomes can also be defined in terms of fixing other endogenous variables: $Y_{i,t+h}(y_j)$, then $Y_{i,t+h} \equiv Y_{i,t+h}(Y_{j,t})$.

Potential Outcomes assumptions

Assumption 1 (Non-anticipating Potential Outcomes)

$Y_t(w_{1:t}, \{w_s\}_{s \geq t+1}) = Y_t(w_{1:t}, \{w'_s\}_{s \geq t+1})$ *a.s.*

Assumption 2 (Sequentially Probabilistic Assignment Process)

$0 < \Pr(W_t = w \mid \mathcal{F}^Y_{t-1}) < 1$ *a.s. for all* $w \in \mathcal{W}$.

Quantile Impulse Response (QIR)

Assumption 3 (Structural Quantile Function (SQF))
Each $Y_{i,t+h}(y_j)$ admits a SQF $q_h(\tau \mid y_j)$, non-decreasing in τ.

- The Quantile Impulse Response is:

$$\mathrm{QIR}_\tau(h) = \frac{\partial q_h(\tau \mid y_j)}{\partial y_j}.$$

More definitions

Mean	Quantile
$\mathrm{IR}(h) = \frac{\partial \mathbb{E}[Y_{i,t+h}(y_j)]}{\partial y_j}$	$\mathrm{QIR}(h)_\tau = \frac{\partial q_h(\tau \vert y_j)}{\partial y_j}$
$\mathrm{cIR}(h) = \frac{\partial \mathbb{E}[Y_{i,t+h}(y_j, x)]}{\partial y_j}$	$\mathrm{cQIR}(h)_\tau = \frac{\partial q_h(\tau \vert y_j, x)}{\partial y_j}$

Quantile Invariance Theorem

If a model has a structural Wold representation with Guassian innovations then all conditional and unconditional QIRs collapse to the mean IR.

Theorem 1 (Quantile Invariance)

If $Y_t = \sum_{j=0}^{\infty} \Psi_j W_{t-j}$ *with* $W_t \overset{iid}{\sim} \mathcal{N}(0, I)$, *then for all* $\tau \in (0,1)$

$$\mathrm{IR}(h) = \mathrm{cIR}(h) = \mathrm{cQIR}(h)_\tau = \mathrm{QIR}(h)_\tau$$

$\rightarrow$ Nontrivial quantile dynamics require departures from linearity, Gaussianity, or covariance stationarity.

Proof

Conditioning in mean versus quantile models

- $\text{cIR}(h) = \text{IR}(h)$ in models with additive separability between treatment and controls, as the expectations operator is linear.
    - In cases where $\text{cIR}(h) \neq \text{IR}(h)$, $\text{IR}(h)$ is not well defined.
- $\text{cQIR}(h) \neq \text{QIR}(h)$ in general even when controls enter additively as the quantile operator is not linear.
- In a large class of models: $\text{cIR}(h) = \text{IR}(h)$ but $\text{cQIR}(h) \neq \text{QIR}(h)$.

Minimal example

Impulse Response identification when shocks are unobservable

Assumption 4 (Independent Assignments)
For all $t \neq s$, $W_t \perp\!\!\!\perp W_s$, *and for all* $i \neq j$, $W_{i,t} \perp\!\!\!\perp W_{j,t}$.

Assumption 5 (Deterministic Potential Outcomes)
Potential outcomes $Y_t(w_{1:t})$ *are deterministic functions of the assignment sequence for all* $t \geq 1$ *and* $w_{1:t} \in \mathcal{W}^t$.

Assumption 6 (Partial Invertibility)
$Y_{j,t} = g_j(W_{j,t}, X_t)$. *The inverse* $W_{j,t} = g_j^{-1}(Y_{j,t}, X_t)$ *exists.*

SVAR(p) satisfies assumptions 1, 2, 4*, 5, 6. details IR identification

Quantile Impulse Response identification

Assumption 7 (Rank Similarity)

For all y_j, y'_j: $\mathbb{P}[Y_{i,t+h}(y_j) \leq q_h(\tau \mid y_j) \mid Y_{j,t}, X_t] = \mathbb{P}[Y_{i,t+h}(y'_j) \leq q_h(\tau \mid y'_j) \mid Y_{j,t}, X_t]$. details

Theorem 2 ($\mathrm{QIR}_\tau(h)$ Identification)

Suppose Assumptions 1 through 7 hold $\forall h$, *then*
$\forall h \in \{0, 1, 2, \ldots, H\}$ *and for each* $\tau \in (0, 1)$*:*
$\mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t}) \mid Y_{j,t}, X_t] = \mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t}) \mid X_t]$,
$\mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t})] = \tau$.

estimation confidence intervals proof

Contrast with QLP: $\mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t}, X_t) \mid Y_{j,t}, X_t] = \tau$.

Key takeaways

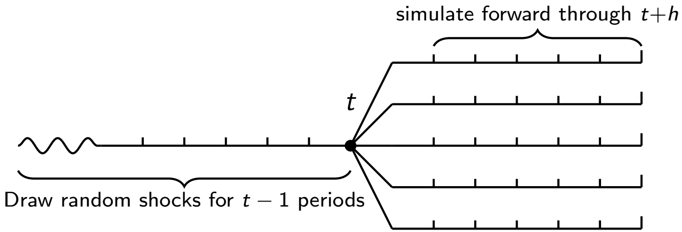
Endogenous volatility SVAR:

- GQLPs with timing restrictions identifies the response of quantiles of growth to a structural shock to financial conditions.

Nonlinear DSGE of Gertler et al. 2019:

- Adding controls to QLP yields misleading conclusions about capital quality shocks' effects on output, even when the shock is observed.
- GQLP capture unconditional effects across all states showing the large asymmetry in response of downside risk.

Simulating Potential Outcomes to Recover the Truth



perturb at t to
a grid of values

SVAR with endogenous volatility

$$\overbrace{Y_{i,t}}^{\text{growth}} = a_{1,11} Y_{i,t-1} + a_{1,12} Y_{j,t-1} + \overbrace{\frac{1+\phi\sqrt{\exp(Y_{j,t-1})}}{1+\phi}}^{\text{stochastic volatility}} W_{i,t}$$

$$\underbrace{Y_{j,t}}_{\text{financial conditions}} + a_{0,22} Y_{i,t} = a_{1,21} Y_{i,t-1} + a_{1,22} Y_{j,t-1} + W_{j,t}$$

parameters

- $W_{i,t}, W_{j,t} \overset{iid}{\sim} N(0,1)$ are unobserved structural shocks.
- If $\phi = 0 \rightarrow$ standard SVAR(1) with timing restrictions.

Can we recover the response of quantiles of $Y_{i,t+h}$ to a structural shock $W_{j,t}$ if we observe only $\{Y_{i,t}, Y_{j,t}\}$?

Recovering the Quantile Impulse Response

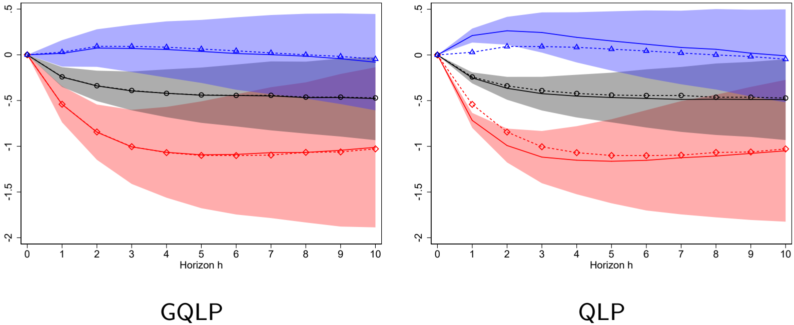


Figure 2: Dashed lines show true QIRs for quantiles $\tau \in \{0.1\diamond, 0.5\circ, 0.9\triangle\}$. Solid lines with shaded areas are average estimates with Monte Carlo standard errors. X-axis is the horizon.

GQLP recovers response of quantiles of $Y_{i,t+h}$ to shock $W_{j,t}$.

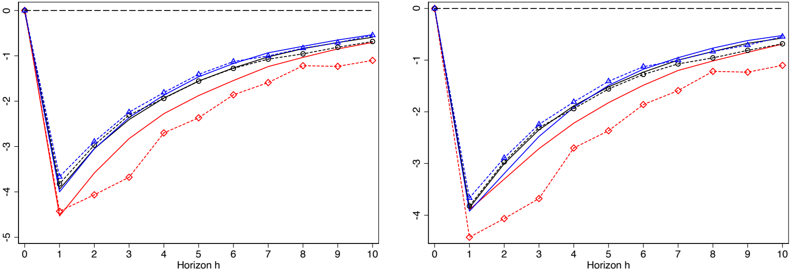
CI coverage SQF

A Macroeconomic Model with Financial Panics (Gertler et al. 2019)

- Nonlinear DSGE model with bankers and households solved non-linearly using a global solution method.
- Bankers operate using equity and household deposits but can divert funds and face bankruptcy.
- Households can manage capital but less efficiently than banks.
- Bank run equilibrium exists when liquidation prices are low enough that banks have negative net worth.
- Series of negative capital quality shocks can bring economy to point where bank runs become possible.

MIT shock SQF

Responses of quantiles of output growth to capital quality shock



GQLP QLP

Figure 3: True QIR (dashed) versus estimated QIR (solid) under the linear specification for quantiles $\tau \in \{0.1\diamond, 0.5\circ, 0.9\triangle\}$. Results from $MC = 1{,}000$ simulations of length $T = 500$). X-axis is the horizon.

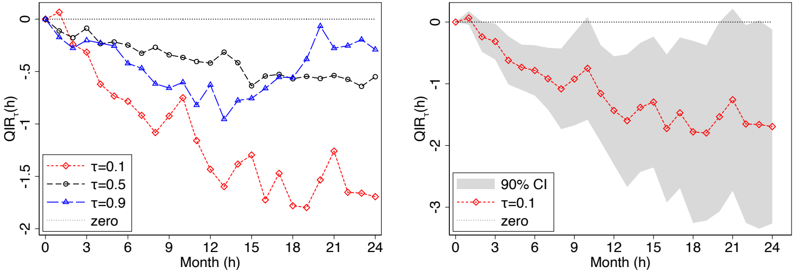
Data

- Monthly US time-series from January 1984 and June 2025.
- $Y_{i,t+h}$ is the h-months cumulative Industrial Production growth.
- Two distinct financial risk shocks $Y_{j,t}$:
  1. Credit risk shock: Δ Excess Bond Premium.
  2. Volatility risk premium: realized - implied volatility of S&P500.
- X_t is a vector of contemporaneous and lagged values of macroeconomic, financial and monetary variables.

Identification assumption

- Timing restriction: financial conditions are affected contemporaneously by macro shocks but respond with lag to monetary policy shocks.
- Financial variables adjust quicker than real variables (Sims 1980; Christiano et al. 1996; Bloom 2009; Gilchrist and Zakrajšek 2012; Chavleishvili and Manganelli 2024).
- I check robustness of results to alternative orderings (financial risk variable ordered first and last). credit volatility

$\text{QIR}_\tau(h)$ to credit risk shock

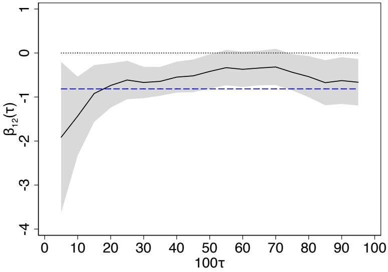


(a) $\tau \in \{0.1, 0.5, 0.9\}$ (b) $\tau = 0.1$ with 90% C.I.

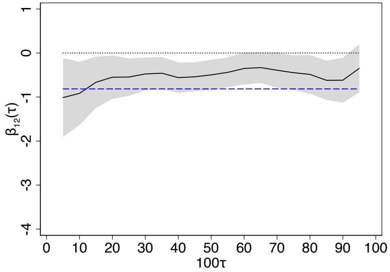
Figure 4: Cumulative loss to Industrial Production growth (in % pts.) from 1 std. dev. credit risk shock. X-axis is the horizon in months.

τ 0.5, 0.9

GQLP vs QLP vs LP at one year horizon - Credit risk



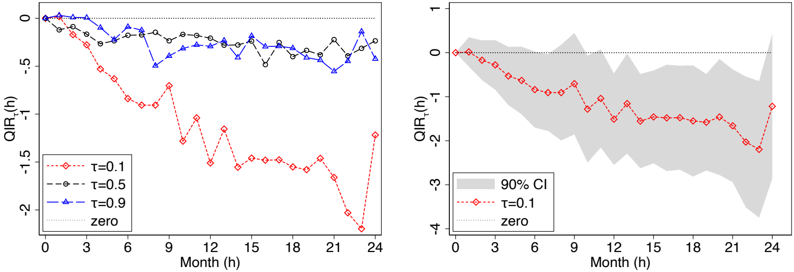
GQLP credit risk



QLP credit risk

Figure 5: Response at one-year horizon, x-axis is the quantile τ (multiplied by 100 for legibility). Shaded area is the 90% CI. Blue dashed line is the response of the mean. All models have the same variables, ordering, and lag length.

$\mathrm{QIR}_\tau(h)$ to volatility risk shock

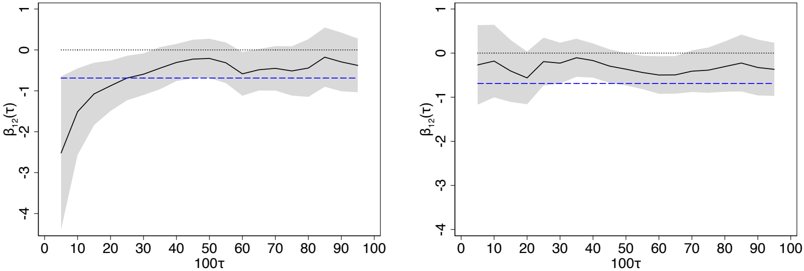


(a) $\tau \in \{0.1, 0.5, 0.9\}$ (b) $\tau = 0.1$ with 90% C.I.

Figure 6: Cumulative loss to Industrial Production growth (in % pts.) from 1 std. dev. volatility risk shock. X-axis is the horizon in months.

τ 0.5, 0.9

GQLP vs QLP vs LP at one year horizon - Volatility risk



GQLP volatility risk QLP volatility risk

Figure 7: Response at one-year horizon, x-axis is the quantile τ (multiplied by 100 for legibility). Shaded area is the 90% CI. Blue dashed line is the response of the mean. All models have the same variables, ordering, and lag length.

GQLP vs QLP - Lag length sensitivity

Quantile	Horizon	Lags	Credit risk QLP	Credit risk GQLP	Volatility risk QLP	Volatility risk GQLP
0.1	12	1	-0.94	-1.37	-0.22	-1.44
		2	-0.92	-1.43	-0.18	-1.51
		3	-0.64	-1.33	-0.03	-1.46
		4	-0.55	-1.33	-0.33	-1.44
	24	1	-1.32	-1.69	-2.04	-1.61
		2	-0.99	-1.69	-1.53	-1.22
		3	-1.54	-1.73	-1.64	-1.02
		4	-1.29	-1.73	-1.22	-1.02

Table 1: Responses of Industrial Production (in % pts.) to a shock that increases credit risk or volatility risk by one standard deviation, computed for two horizons $h \in \{12, 24\}$ and quantile $\tau \in \{0.1\}$. Lags refers to the number of lags included as covariates (2 lags is the baseline specification).

Comparison with Previous Literature

- Prior findings based on QLP with NFCI as a financial risk measure:
  - Adrian et al. 2019: $\text{QIR}_{0.1} = \mathbf{-1.75}$pp vs. $\text{QIR}_{0.5} = \mathbf{-0.75}$pp.
  - Ruzicka 2021: $\text{QIR}_{0.1} = \mathbf{-2.5}$pp vs. $\text{QIR}_{0.5} = \mathbf{-1.5}$pp.
- Downside to median response asymmetry: $\frac{\text{QIR}_{0.1}}{\text{QIR}_{0.5}}$:
  - Previous Literature: Ratio of $\approx \mathbf{2}$.
- I find a significantly *stronger* degree of asymmetry:
  - Credit Risk Shock: Ratio of **2.8**
  - Volatility Risk Shock: Ratio of **4.1**

Conclusion

- GQLP enables causal inference on *unconditional* quantiles while using controls.
  - Unconditionally low growth $\rightarrow$ actual downturns.
- Additional advantage of GQLP over existing alternatives:
  - QIRs defined as responses to one-off shocks.
  - Can accommodate nonlinear functional forms.
  - Naturally extends to instrumental variable designs.
- Empirical findings:
  - 2pp loss at lower tail vs 0.5pp at median.
  - Stabilizing financial conditions prevents recessions without sacrificing expansion growth.

Future Research

- Methodological extensions:
  - Incorporate instrumental variables identification.
  - Apply smoothing techniques to GQLP following Barnichon and Brownlees (2019) and Ruzicka (2021).
- Other applications:
  - Inflation-at-risk: How do policy shocks affect inflation in high vs low inflation environments?
  - Fiscal multiplier: Do government spending shocks have different effects during expansions vs contractions?

References I

- Adrian, Tobias, Nina Boyarchenko, and Domenico Giannone (2019). "Vulnerable Growth". In: *American Economic Review* 109.4, pp. 1263–89.
- Adrian, Tobias, Federico Grinberg, Nellie Liang, Sheheryar Malik, and Jie Yu (2022). "The Term Structure of Growth-at-Risk". In: *American Economic Journal: Macroeconomics* 14.3, pp. 283–323.
- Barnichon, Regis and Christian Brownlees (2019). "Impulse response estimation by smooth local projections". In: *Review of Economics and Statistics* 101 (3).
- Bloom, Nicholas (2009). "The Impact of Uncertainty Shocks". In: *Econometrica* 77.3, pp. 623–685.
- Brownlees, Christian and André B.M. Souza (2021). "Backtesting global Growth-at-Risk". In: *Journal of Monetary Economics* 118, pp. 312–330.

References II

- Chavleishvili, Sulkhan and Simone Manganelli (2024). "Forecasting and stress testing with quantile vector autoregression". In: *Journal of Applied Econometrics* 39.1, pp. 66–85.
- Christiano, Lawrence, Martin Eichenbaum, and Charles Evans (1996). "The Effects of Monetary Policy Shocks: Evidence from the Flow of Funds". In: *The Review of Economics and Statistics* 78.1, pp. 16–34.
- Gertler, Mark, Nobuhiro Kiyotaki, and Andrea Prestipino (2019). "A Macroeconomic Model with Financial Panics". In: *The Review of Economic Studies* 87.1, pp. 240–288.
- Gilchrist, Simon and Egon Zakrajšek (2012). "Credit Spreads and Business Cycle Fluctuations". In: *American Economic Review* 102.4, pp. 1692–1720.

References III

Jordá, Ó. (2005). "Estimation and inference of impulse responses by local projections". In: *American Economic Review* 95 (1).

Linnemann, Ludger and Roland Winkler (2016). "Estimating nonlinear effects of fiscal policy using quantile regression methods". In: *Oxford Economic Papers* 68 (4).

Loria, Francesca, Christian Matthes, and Donghai Zhang (2025). "Assessing Macroeconomic Tail Risk". In: *The Economic Journal* 135.665, pp. 264–284.

Plagborg-Møller, Mikkel and Christian K. Wolf (2021). "Local Projections and VARs Estimate the Same Impulse Responses". In: *Econometrica* 89 (2).

Powell, David (2020). "Quantile Treatment Effects in the Presence of Covariates". In: *The Review of Economics and Statistics* 102.5, pp. 994–1005.

References IV

Rambachan, Ashesh and Neil Shephard (2021). "When do common time series estimands have nonparametric causal meaning?" In: *Manuscript, Harvard University*.

Ruzicka, Josef (2021). "Quantile local projections: Identification, smooth estimation, and inference". In: *Universidad Carlos III de Madrid*.

Sims, Christopher A (1980). "Macroeconomics and Reality". In: *Econometrica* 48.1, pp. 1–48.

Appendix

SVAR & Potential Outcomes (Rambachan and Shephard 2021)

SVAR(1) model assumes that the potential outcome process satisfies:

$$A_0 Y_t(w_{1:t}) = w_t + A_1 Y_{t-1}(w_{1:t-1}).$$

Then $A_0(I - \Phi_1 L) Y_t(w_{1:t}) = w_t$, where L is the lag operator and $\Phi_1 = A_0^{-1} A_1$. So

$$Y_t(w_{1:t}) = A_0^{-1} w_t + \Phi_1 Y_{t-1}(w_{1:t-1}),$$

which in turn implies that the potential outcome process also has an SVMA model representation:

$$Y_t(w_{1:t}) = A_0^{-1} w_t + \Phi_1 A_0^{-1} w_{t-1} + \cdots + \Phi_1^{t-1} A_0^{-1} w_1 + \Phi_1^t A_0^{-1} Y_0.$$

In this case, the h-period ahead average treatment effect is

$$\mathbb{E}\left[Y_{t+h}(w) - Y_{t+h}(w')\right] = \Phi_1^h A_0^{-1}(w - w').$$

Go back

Mean Impulse Response identification

Theorem 3 (IR(h) Identification)

Under Assumptions 1, 2, 4, 5, 6:

$$\frac{\partial}{\partial y_j}\mathbb{E}[Y_{i,t+h} \mid Y_{j,t} = y_j, X_t] = \mathbb{E}\left[\overbrace{\underbrace{\frac{\partial Y_{i,t+h}(w_j)}{\partial w_j}}_{\textit{SVAR IR}} \cdot \underbrace{\frac{\partial g_j^{-1}(y_j, X_t)}{\partial y_j}}_{\textit{normalization}}}^{=\text{IR(h)}} \,\middle|\, X_t\right].$$

proof Go back

Normalization does not depend on the horizon nor the dependent variable and is a constant in linear settings. In a SVAR it equals to 1, and in Local Projections it can be estimated (Plagborg-Møller and Wolf 2021).

Proof - Quantile Invariance I

$Y_t \in \mathbb{R}^k$ admits a structural Wold representation

$$Y_t = \sum_{\ell=0}^{\infty} \Psi_\ell W_{t-\ell}, \qquad \sum_{\ell=0}^{\infty} \|\Psi_\ell\| < \infty,$$

where $\{W_t\}_{t\in\mathbb{Z}}$ are i.i.d. $\mathcal{N}(0, I_k)$. Fix $i, j \in \{1, \ldots, k\}$ and a horizon $h \geq 0$. Write the i-th component at horizon h as

$$Y_{i,t+h} = [\Psi_h]_{i,j}\, W_{j,t} + R_{i,t+h},$$

where

$$R_{i,t+h} := \sum_{m\neq j} [\Psi_h]_{i,m} W_{m,t} + \sum_{\ell\neq h} \sum_{m=1}^{k} [\Psi_\ell]_{i,m} W_{m,t+h-\ell}$$

Go back

Proof - Quantile Invariance II

Because the innovations are i.i.d. Gaussian, $W_{j,t}$ is independent of $R_{i,t+h}$ (uncorrelated jointly Gaussian variables are independent). Applying the definition of potential outcomes yields:

$$Y_{i,t+h}(w_j) = [\Psi_h]_{i,j}\, w_j + R_{i,t+h}.$$

Mean Impulse Response (IR):

Taking expectations yields:

$$\mathbb{E}[Y_{i,t+h}(w_j)] = \mathbb{E}[R_{i,t+h} + [\Psi_h]_{i,j} w_j] = \mathbb{E}[R_{i,t+h}] + [\Psi_h]_{i,j}\, w_j.$$

Differentiating w.r.t. w_j yields:

$$\mathrm{IR}(h) = \frac{\partial\, \mathbb{E}[Y_{i,t+h}(w_j)]}{\partial w_j} = [\Psi_h]_{i,j}.$$

Go back

Proof - Quantile Invariance III I

Quantile Impulse Response (QIR):

$$\begin{aligned} q_h(\tau \mid w_j) &\equiv q_{Y_{i,t+h}(w_j)}(\tau \mid W_{t,j} = w_j) \\ &= q_{R_{i,t+h}+[\Psi_h]_{i,j} w_j}(\tau \mid W_{t,j} = w_j) \\ &= q_{R_{i,t+h}}(\tau \mid W_{t,j} = w_j) + [\Psi_h]_{i,j} w_j \\ &= q_{R_{i,t+h}}(\tau) + [\Psi_h]_{i,j} w_j \end{aligned}$$

Differentiating w.r.t. w_j yields:

$$\mathrm{QIR}_\tau(h) = \frac{\partial\, q_h(\tau \mid W_{j,t} = w_j)}{\partial w_j} = [\Psi_h]_{i,j}.$$

Therefore $\mathrm{QIR}_\tau(h) = [\Psi_h]_{i,j}$ for all $\tau \in (0,1)$. Go back

Proof - Quantile Invariance IV I

Conditional Quantile Impulse Response (cQIR):
Let X_t be any vector measurable w.r.t. past and current observables.

$$
\begin{aligned}
q_h(\tau \mid w_j, x) &\equiv q_{Y_{i,t+h}(w_j,x)}(\tau \mid W_{j,t} = w_j,\, X_t = x) \\
&= q_{R_{i,t+h}+[\Psi_h]_{i,j}\, w_j}(\tau \mid W_{j,t} = w_j,\, X_t = x) \\
&= q_{R_{i,t+h}}(\tau \mid W_{j,t} = w_j,\, X_t = x) + [\Psi_h]_{i,j}\, w_j \\
&= q_{R_{i,t+h}}(\tau \mid X_t = x) + [\Psi_h]_{i,j}\, w_j
\end{aligned}
$$

Differentiating w.r.t. w_j yields:

$$
\mathrm{cQIR}_\tau(h) = \frac{\partial\, q_h(\tau \mid W_{j,t} = w_j, , X_t = x)}{\partial w_j} = [\Psi_h]_{i,j}.
$$

Therefore $\mathrm{cQIR}_\tau(h) = [\Psi_h]_{i,j}$ for all $\tau \in (0,1)$. Go back

Rank Similarity Assumption

- Weaker version of *Rank Invariance* $\rightarrow$ rank is identical across treatment states.
- If we know the current treatment and controls, whether a time period would have a high-rank or low-rank outcome doesn't depend on which treatment value we're considering.
- If economic conditions suggest a period would experience an above-median outcome given one policy shock value, those same conditions suggest it would also experience an above-median outcome for a different policy shock value (though the median levels themselves differ).

Go back

Proof - IR identification I

Define the covariate set: $X_t \equiv \{Y_{-j,t}, \mathcal{F}_{t-1}^{Y}\}$. Using the shortcut and conditional independence, we write:

$$\begin{aligned}\mathbb{E}[Y_{i,t+h}\mathbf{1}\{Y_{j,t}=y_j\} \mid X_t] &= \mathbb{E}[Y_{i,t+h}(W_{1:t+h})\mathbf{1}\{Y_{j,t}=y_j\} \mid X_t] \\ &= \mathbb{E}[Y_{i,t+h}(g_j^{-1}(y_j, X_t))\mathbf{1}\{Y_{j,t}=y_j\} \mid X_t].\end{aligned}$$

Expand via expectation and covariance:

$$\begin{aligned}&= \mathbb{E}[Y_{i,t+h}(g_j^{-1}(y_j, X_t)) \mid X_t]\,\mathbb{E}[\mathbf{1}\{Y_{j,t}=y_j\} \mid X_t] \\ &\quad + \operatorname{Cov}\big(Y_{i,t+h}(g_j^{-1}(y_j, X_t)), \mathbf{1}\{Y_{j,t}=y_j\} \mid X_t\big).\end{aligned}$$

Under conditional random assignment, the covariance is zero

$$\begin{aligned}&\mathbb{E}[Y_{i,t+h}\mathbf{1}\{Y_{j,t}=y_j\} \mid X_t] \\ &= \mathbb{E}[Y_{i,t+h}(g_j^{-1}(y_j, X_t)) \mid X_t]\,\mathbb{E}[\mathbf{1}\{Y_{j,t}=y_j\} \mid X_t].\end{aligned}$$

Proof - IR identification II

Use the identity $\mathbb{E}[A \mid B] = \frac{\mathbb{E}[A\mathbf{1}\{B\}]}{\mathbb{E}[\mathbf{1}\{B\}]}$ to obtain:

$$\mathbb{E}[Y_{i,t+h} \mid Y_{j,t} = y_j, X_t] = \mathbb{E}\big[Y_{i,t+h}(g_j^{-1}(y_j, X_t)) \mid X_t\big].$$

Taking partial derivative w.r.t. y_j and exchanging derivative/expectation:

$$\begin{aligned}\frac{\partial}{\partial y_j}\mathbb{E}[Y_{i,t+h} \mid Y_{j,t} = y_j, X_t] &= \mathbb{E}\Big[\frac{\partial}{\partial y_j} Y_{i,t+h}(g_j^{-1}(y_j, X_t)) \mid X_t\Big] \\ &= \mathbb{E}\Big[\frac{\partial Y_{i,t+h}(w_j)}{\partial w_j}\,\frac{\partial g_j^{-1}(y_j, X_t)}{\partial y_j} \mid X_t\Big].\end{aligned}$$

Go back

Proof - QIR identification I

We restate Theorem 1 from Powell 2020 in our settng. First, we want to show: $\mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t}) \mid Y_{j,t}, X_t] = \mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t}) \mid X_t]$. Evaluating the left hand side of the equality we have:

$$\begin{aligned}
\mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t}) \mid Y_{j,t}, X_t] &= \mathbb{P}[Y_{i,t+h}(Y_{j,t}) \leq q_h(\tau \mid Y_{j,t}) \mid Y_{j,t}, X_t] \\
&= \mathbb{P}[Y_{i,t+h}(y_j) \leq q_h(\tau \mid y_j) \mid Y_{j,t}, X_t] \\
&= \mathbb{P}[Y_{i,t+h}(y_j) \leq q_h(\tau \mid y_j) \mid X_t].
\end{aligned}$$

The first equality sign follows from the definition of a potential outcome. The second equality sign comes from the rank similarity assumption which must hold for all y_j, y_j' and thus also for $d = Y_{j,t}$. The third

Proof - QIR identification II

equality sign follows from the conditional independence of Y_j. Evaluating the right hand side of the equality we have:

$$
\begin{aligned}
&\mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t}) \mid X_t] = \mathbb{P}[Y_{i,t+h}(Y_{j,t}) \leq q_h(\tau \mid Y_{j,t}) \mid X_t] \\
&= \int \mathbb{P}[Y_{i,t+h}(Y_{j,t}) \leq q_h(\tau \mid Y_{j,t}) \mid X_t, Y_{j,t}] d\mathbb{P}(Y_{j,t} \mid X_t) \\
&= \int \mathbb{P}[Y_{i,t+h}(y_j) \leq q_h(\tau \mid y_j) \mid Y_{j,t}, X_t] d\mathbb{P}(Y_{j,t} \mid X_t) \\
&= \mathbb{P}[Y_{i,t+h}(y_j) \leq q_h(\tau \mid y_j) \mid X_t].
\end{aligned}
$$

The first equality follows from the definition of a potential outcome. The third equality follows from the rank similarity assumption. The second and fourth equality follow directly from properties of marginal probability functions.

Proof - QIR identification III

Now we want to show: $\mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t})] = \tau$.

$$
\begin{aligned}
\mathbb{P}[Y_{i,t+h} \leq q_h(\tau \mid Y_{j,t})] &= \int \mathbb{P}[Y_{i,t+h}(Y_{j,t}) \leq q_h(\tau \mid Y_{j,t}) \mid X_t, Y_{j,t}] d\mathbb{P}(X_t, Y_{j,t}) \\
&= \int \mathbb{P}[Y_{i,t+h}(y_j) \leq q_h(\tau \mid y_j) \mid X_t, Y_{j,t}] d\mathbb{P}(X_t, Y_{j,t}) \\
&= \mathbb{P}[Y_{i,t+h}(y_j) \leq q_h(\tau \mid y_j)] \\
&= \tau
\end{aligned}
$$

The second equality follows from the rank similarity assumption. The fourth equality follows from SQF assumption.

Go back

Estimation is done separately for each h and for each τ:

1. Postulate a candidate $\tilde{\beta}_h(\tau)$. For each candidate $\tilde{\beta}_h(\tau)$ there exists an intercept $\tilde{\alpha}_h(\tau)$ such that $\mathbb{P}(Y_{i,t+h} \leq \tilde{\alpha}_h(\tau) + \tilde{\beta}_h(\tau) Y_{j,t}) = \tau$. We need to search over the slope coefficients only.
2. Given the pair $(\tilde{\alpha}_h(\tau), \tilde{\beta}_h(\tau))$, estimate a linear probability model (or Logit/Probit) for the event that $Y_{i,t+h} \leq \tilde{\alpha}_h(\tau) + \tilde{\beta}_h(\tau) Y_{j,t}$ as a function of controls X_t. Save the predicted probabilities as $\hat{\tau}_{X_t}$.
3. $\hat{\beta}_h(\tau) = \text{argmin}_{\tilde{\beta}_h(\tau)} g' A g$, where $g = \frac{1}{T} \sum_{t=1}^{T} Y_{j,t}[\mathbf{1}\{Y_{i,t+h} \leq \tilde{\alpha}_h(\tau) + Y_{j,t}\tilde{\beta}_h(\tau)\} - \hat{\tau}_{X_t}]$. $A = [\hat{E}(gg')]^{-1}$ is the optimal GMM weighting matrix constructed using starting values from standard quantile regression of $Y_{i,t+h}$ on $Y_{j,t}$.

go back

Block of Block Bootstrap Confidence Intervals

- Re-sampling the data by randomly drawing blocks of m consecutive observations, using blocks which start at indexes $1, \ldots, T-m+1$, before re-estimating the model B times using these pseudo-samples.
- The confidence intervals are then based on the distribution of the estimated parameters across the B repetitions of the procedure.
- Re-sampling blocks of observations preserves the temporal dependence in the data.

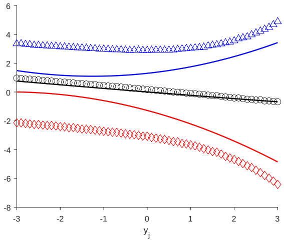
go back

Monte Carlo DGP Parameters

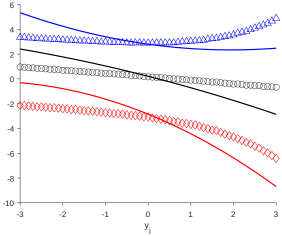
parameter	$a_{1,11}$	$a_{1,22}$	$a_{1,12}$	$a_{1,21}$	$a_{0,22}$	ϕ
value	0.5	-0.1	-0.25	-0.1	-0.2	1

Table 2: Model parameters used in the simulation.

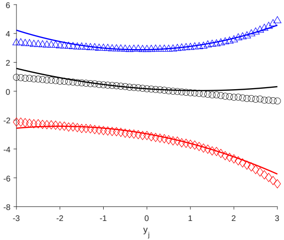
go back



QLP no controls



QLP with controls



GQLP

Simulation results for $q_1(\tau \mid y_j)$ from 1,000 simulations of length 500. $\tau \in \{0.1\diamond, 0.5\circ, 0.9\triangle\}$ show the true SQF. Solid lines show the estimated SQF using a quadratic specification. go back

CI Coverage - Monte Carlo results

Quantile	Horizon	Nominal Level 68 %	90 %	95 %
0.1	1	71%	89%	94%
	5	68%	91%	96%
	10	70%	89%	94%
0.5	1	67%	92%	95%
	5	69%	90%	95%
	10	70%	91%	95%
0.9	1	68%	88%	93%
	5	70%	89%	96%
	10	72%	91%	96%
Average coverage		69.4%	90.0%	95.0%

Table 3: The coverage was computed in a Monte Carlo simulation using the endogenous volatility SVAR model with 500 repetitions of length $T = 500$ and $B = 1000$ Bootstrap repetitions. The block size used was $m = 7$. go back

MIT shock in nonlinear DSGE model (Gertler et al. 2019)

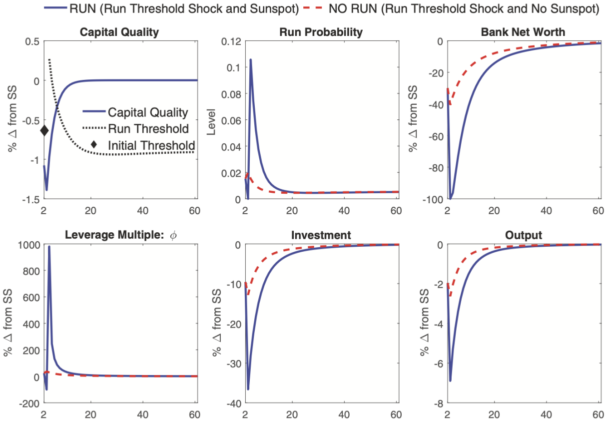


Figure 9: From Gertler et al. (2019). Response of the economy to a sequence of three small negative capital quality shocks combined with a sunspot that triggers a bank run. The plot starts in period 2, economy is in steady state in period 0 then it experiences the three shocks and no shocks thereafter. go back

SQF in nonlinear DSGE model (Gertler et al. 2019)

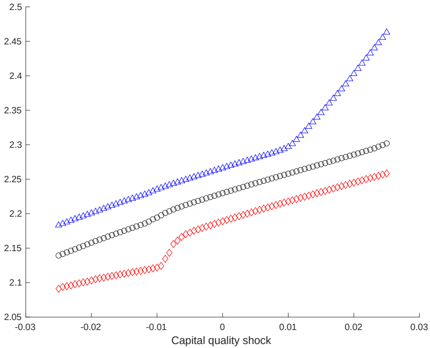
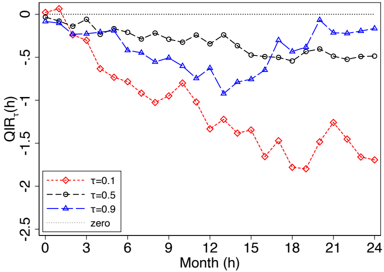
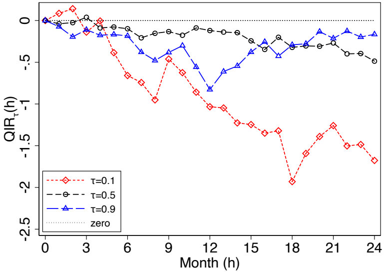


Figure 10: Structural quantile function of output to capital quality shocks at horizon 1, plotted for quantiles $\tau \in \{0.1\diamond, 0.5\circ, 0.9\triangle\}$. X-axis shows the capital quality shocks (the variance of the capital quality shocks is $\sigma = 0.005$), output at horizon 1 is on the vertical axis. The results are by simulating the Gertler et al. (2019) model for each shock grid point over $MC = 1{,}000$ simulation repetitions. go back

Robustness to alternative timing restrictions - Credit risk



(a) Credit risk ordered first

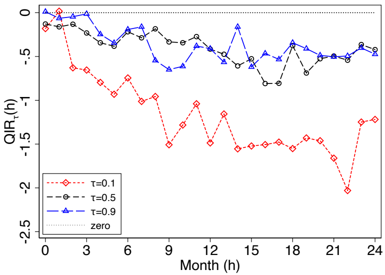


(b) Credit risk ordered last

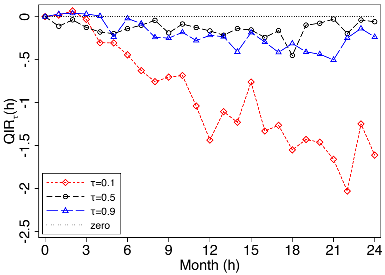
Figure 11: Cumulative loss to Industrial Production growth (in % pts.) from 1 std. dev. credit risk shock. X-axis is the horizon in months.

Go back

Robustness to alternative timing restrictions - Volatility risk



(a) Volatility risk ordered first

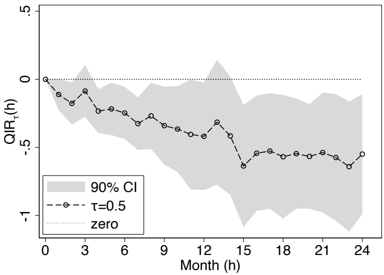


(b) Volatility risk ordered last

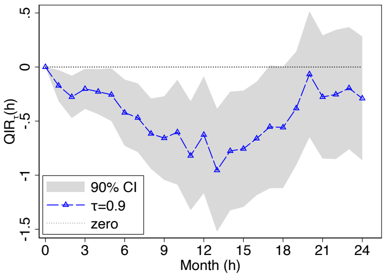
Figure 12: Cumulative loss to Industrial Production growth (in % pts.) from 1 std. dev. volatility risk shock. X-axis is the horizon in months.

Go back

$\mathrm{QIR}_\tau(h)$ to credit risk shock



(a) $\tau = 0.5$ with 90% C.I.

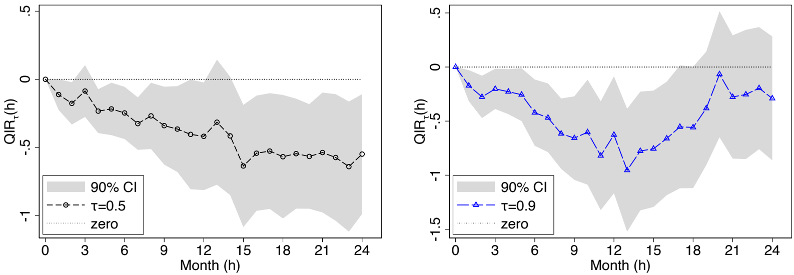


(b) $\tau = 0.9$ with 90% C.I.

Figure 13: Cumulative loss to Industrial Production growth (in % pts.) from 1 std. dev. credit risk shock.

go back

$\text{QIR}_\tau(h)$ to volatility risk premium



(a) $\tau = 0.5$ with 90% C.I.

(b) $\tau = 0.9$ with 90% C.I.

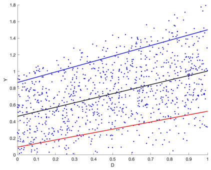
Figure 14: Cumulative loss to Industrial Production growth (in % pts.) from 1 std. dev. volatility risk premium.

go back

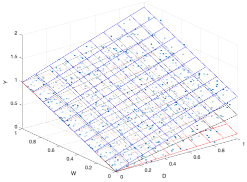
No Frisch-Wough-Lovell theorem for Quantile Regression

$$Y = X_1 W + X_2, \text{where } X_1, X_2, W \overset{iid}{\sim} \text{Uniform}[0,1].$$

- $\mathbb{E}[Y \mid x_1] = x_1 \mathbb{E}[W] + \mathbb{E}[X_2], \quad \mathbb{E}[Y \mid x_1, x_2] = x_1 \mathbb{E}[W] + x_2.$
  $\rightarrow \text{IR} = \text{cIR} = \mathbb{E}[W].$
- $q_Y(\tau \mid x_1, x_2) = x_1 q_W(\tau) + x_2$, but $q_Y(\tau \mid x_1) = q_{x_1 W + X_2}(\tau)$ is not separable.
  $\rightarrow \text{QIR} = \frac{\partial q_{x_1 W + X_2}(\tau)}{\partial x_1} \neq \text{cQIR} = q_W(\tau).$



$\alpha(\tau) + \beta(\tau) X_1$



$\alpha(\tau) + \beta_1(\tau) X_1 + \beta_2(\tau) X_2$

τ	$\beta(\tau)$	$\beta_1(\tau)$
0.1	0.35	0.1
0.5	0.5	0.5
0.9	0.65	0.9

QR coefficients

go back